Number Systems

- Readings: 3-3.3.3, 3.3.5
- Problem: Implement simple pocket calculator
- Need: Display, adders & subtractors, inputs
  - Display: Seven segment displays
  - Inputs: Switches
- Missing: Way to implement numbers in binary
- Approach: From decimal to binary numbers
  (and back)

Arithmetic Operations

Decimal:

$$\begin{array}{r} 5\ 7\ 8\ 9\ 2 \\ +\ 7\ 8\ 9\ 5\ 6 \\ \hline \end{array}$$

Binary:

ADDER

HALF-ADDER

$$\begin{array}{r} 1\ 0\ 1\ 0\ 1\ 1\ 1 \\ +\ 0\ 1\ 0\ 0\ 1\ 0\ 1 \\ \hline 1\ 1\ 1\ 1\ 1\ 0\ 0 \end{array}$$

Decimal:

$$\begin{array}{r} 5\ 7\ 8\ 9\ 2 \\ -\ 3\ 2\ 9\ 4\ 6 \\ \hline \end{array}$$

Binary:

$$\begin{array}{r} 1\ 0\ 1\ 0\ 0\ 1\ 1\ 0 \\ -\ 0\ 0\ 1\ 1\ 0\ 1\ 1\ 1 \\ \hline 0\ 1\ 1\ 0\ 1\ 1\ 1\ 1 \end{array}$$

Arithmetic Operations (cont.)

Decimal:

```
   2 0 1
 * 2 1 4
 -------
```

Binary:

```
        1 0 0 1
      * 1 0 1 1
      ---------
        1 0 0 1
      1 0 0 1 -
    0 0 0 0 - -
  1 0 0 1 - - -
  -------------
  1 1 0 0 0 1 1
```

Half Adder

Ai	Bi	Carry	Sum
0	0	0	0
0	1	0	1
1	0	0	1
1	1	1	0

AND XOR

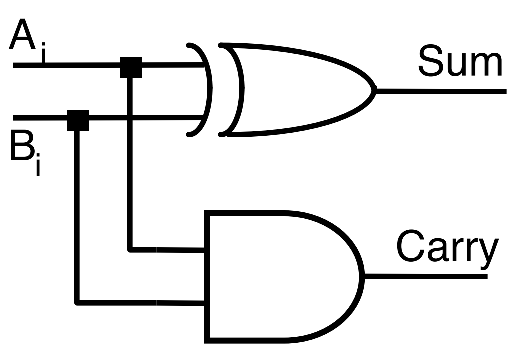
Bi \ Ai		Ai
	0	0
Bi	0	1

Carry = Ai Bi

Bi \ Ai		Ai
	0	1
Bi	1	0

Sum = $\overline{\text{Ai}}$ Bi + Ai $\overline{\text{Bi}}$

= Ai $\oplus$ Bi



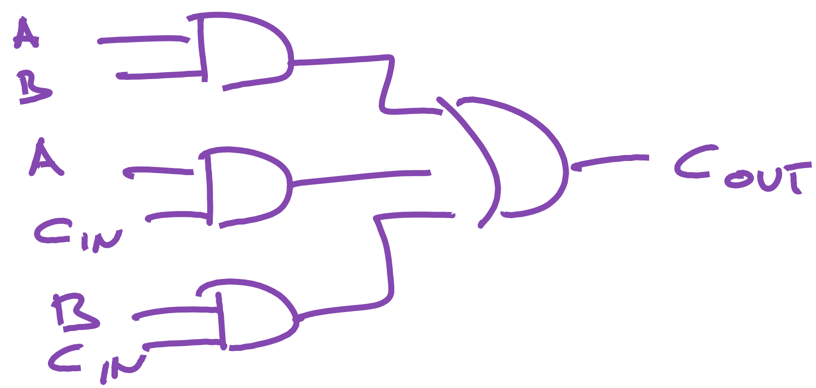
Half-adder Schematic

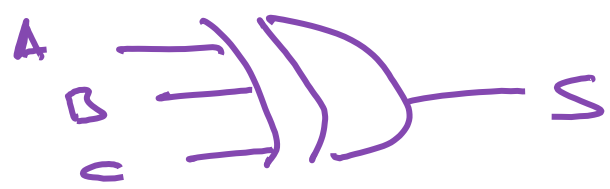
Full Adder

A	B	CI	CO	S
0	0	0	0	0
0	0	1	0	1
0	1	0	0	1
0	1	1	1	0
1	0	0	0	1
1	0	1	1	0
1	1	0	1	0
1	1	1	1	1

$$C_{OUT} = AB + AC_{IN} + BC_{IN}$$

$$S = A \oplus B \oplus C$$



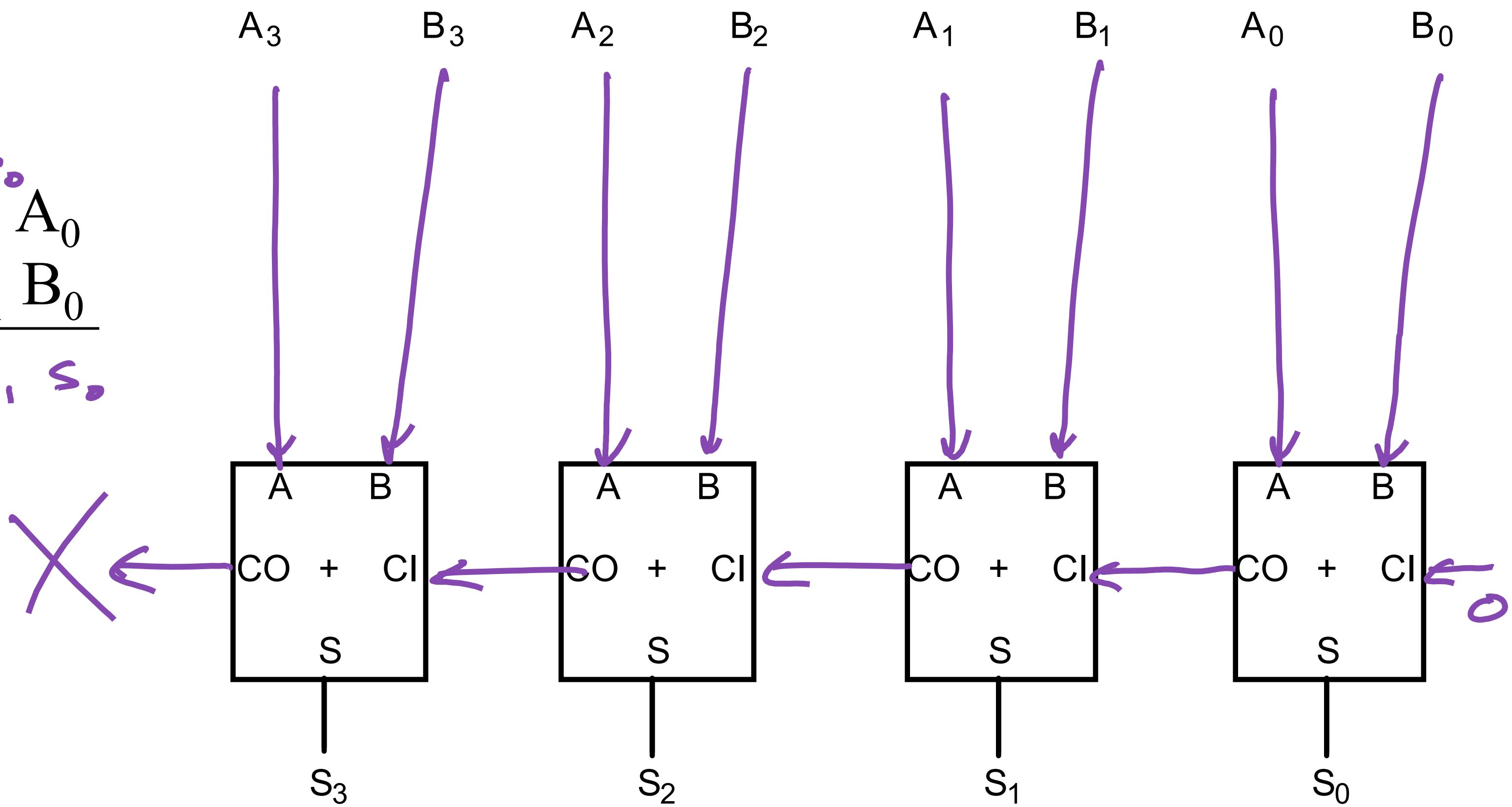


Full Adder Implementation

Multi-Bit Addition

"RIPPLE - CARRY ADDER"

$$
\begin{array}{rcccc}
 & \cancel{c_3} & c_2 & c_1 & c_0 \\
 & A_3 & A_2 & A_1 & A_0 \\
+ & B_3 & B_2 & B_1 & B_0 \\
\hline
 & s_3 & s_2 & s_1 & s_0
\end{array}
$$



Multi-Bit Addition in Verilog, Parameters

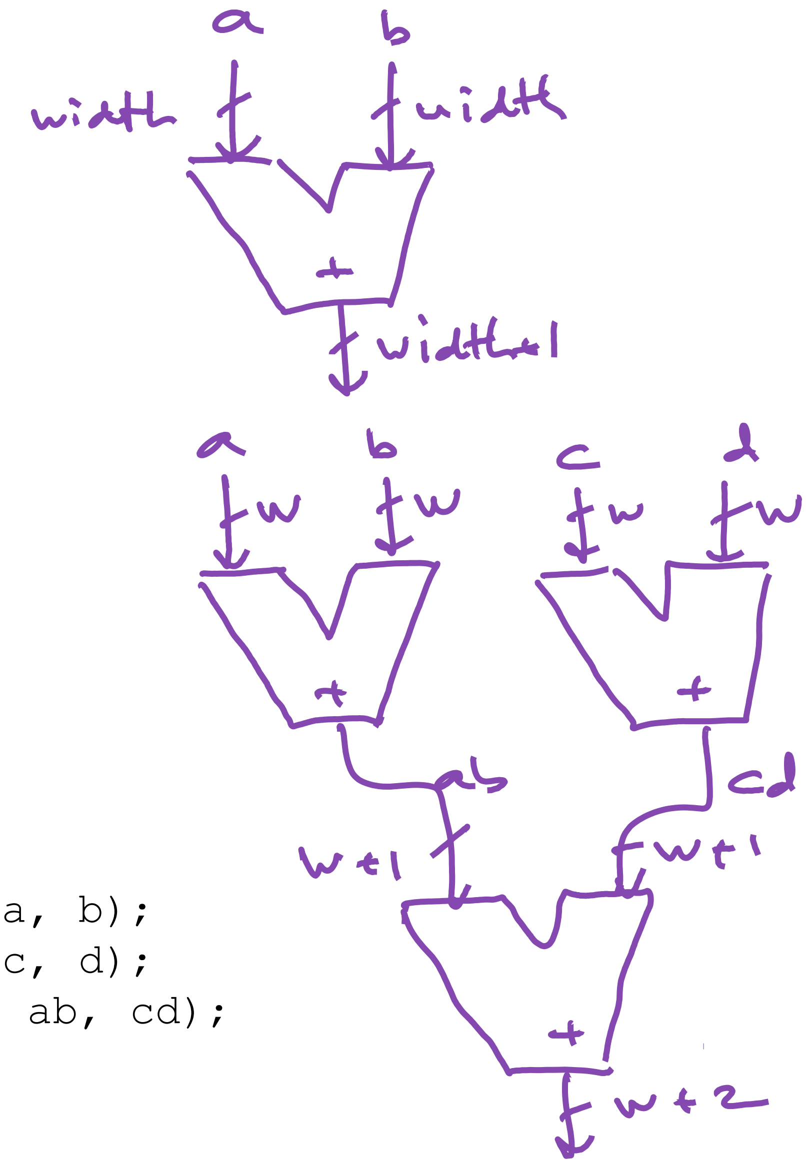
```
module uadd #(parameter WIDTH=8)
   (out, a, b);
  output reg [WIDTH:0] out;
  input      [WIDTH-1:0] a, b;

  always @(*) begin
    out = a + b;
  end
endmodule

module add4 #(parameter W=22)
   (out, a, b, c, d);
  output [W+1:0] out;
  input  [W-1:0] a, b, c, d;

  wire [W:0] ab, cd;

  uadd #(.WIDTH(W))   u_ab   (ab, a, b);
  uadd #(.WIDTH(W))   u_cd   (cd, c, d);
  uadd #(.WIDTH(W+1)) u_abcd (out, ab, cd);
endmodule
```



Negative Numbers

- Need an efficient way to represent negative numbers in binary
  - Both positive & negative numbers will be strings of bits
  - Use fixed-width formats (4-bit, 16-bit, etc.)
- Must provide efficient mathematical operations
  - Addition & subtraction with potentially mixed signs
  - Negation (multiply by -1)

$0 \rightarrow +$

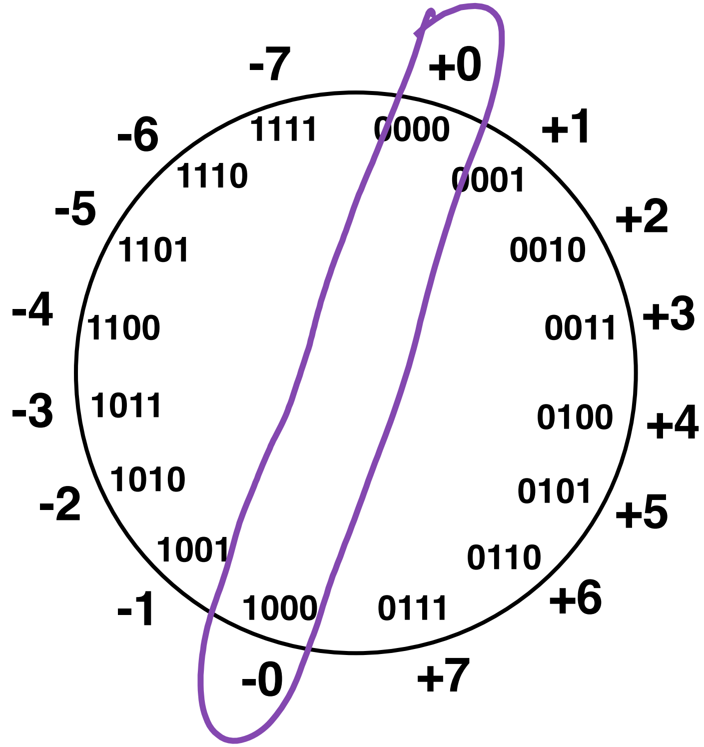
$1 \rightarrow -$

$$\begin{array}{c|ccccc} 1 & 0 & 0 & 0 & 0 & 1 \\ S & M_4 & & & & M_0 \end{array}$$

SIGN

MAGNITUDE

Sign/Magnitude Representation



0 100 = + 4

1 100 = - 4

High order bit is sign: 0 = positive (or zero), 1 = negative

Three low order bits is the magnitude: 0 (000) thru 7 (111)

Number range for n bits = +/-2^{n-1} -1

Representations for 0:

Sign/Magnitude Addition

SIGNS ARE THE SAME: ADD MAGNITUDES, KEEP SIGN

SIGNS ARE DIFFERENT: SUBTRACT SMALLER FROM BIGGER MAG
KEEP SIGN OF BIGGER #

```
    0 | 0 1 0   (+2)
+   0 | 1 0 0   (+4)
-------------
    0   1 1 0   +6
```

```
    1 | 0 1 0   (-2)
+   1 | 1 0 0   (-4)
-------------
    1   1 1 0   -6
```

```
    0 | 0 1 0   (+2)        100
+   1 | 1 0 0   ( -4)      -010
-------------              ----
    1   0 1 0   -2          010
```

```
    1 | 0 1 0   ( -2)       100
+   0 | 1 0 0   (+4)       -010
-------------
    0   0 1 0   +2
```

Bottom line: Basic mathematics are too complex in Sign/Magnitude

Idea: Pick negatives so that addition works

- Let -1 = 0 - (+1):

$$
\begin{array}{rccccl}
1 & 0 & 0 & 0 & 0 & (\ 0) \\
- & 0 & 0 & 0 & 1 & (+1) \\
\hline
 & 1 & 1 & 1 & 1 &
\end{array}
$$

- Does addition work?

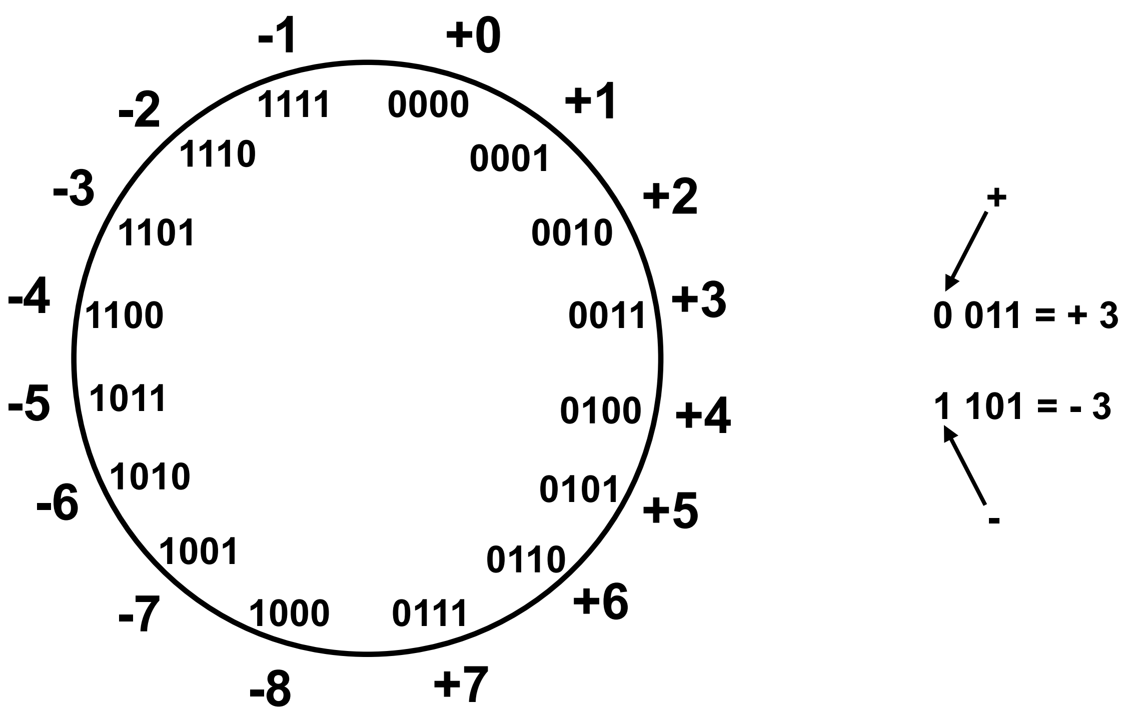
$$
\begin{array}{rccccl}
 & 0 & 0 & 1 & 0 & (+2) \\
+ & 1 & 1 & 1 & 1 & (\ -1) \\
\hline
 & 0 & 0 & 0 & 1 &
\end{array}
$$

- Result: Two's Complement Numbers

FOR $0 \le b \le 2^n - 1$ $\quad -b$ IS REPRESENTED BY $2^n - b$

Two’s Complement

- Only one representation for 0
- One more negative number than positive number
- Fixed width format for both pos. & neg. numbers



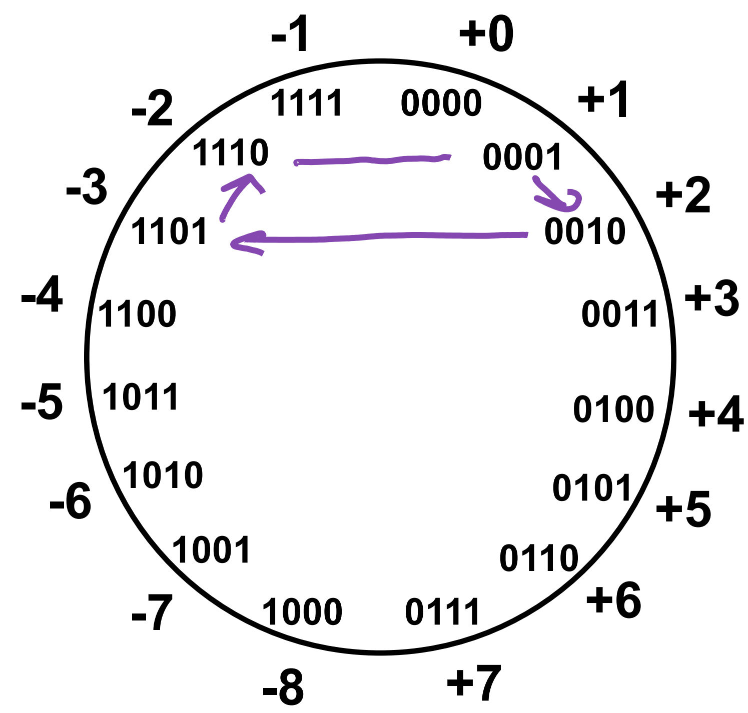
Negating in Two's Complement

- Flip bits & Add 1
- Negate $(0010)_2$ (+2)

$-2 = -0010 = 1101 + 1 = 1110$

- Negate $(1110)_2$ (-2)

$-1110 = 0001 + 1 = 0010$



1

Addition in Two's Complement

$$
\begin{array}{rrrrrl}
 & 0 & 0 & 1 & 0 & (+2) \\
+ & 0 & 1 & 0 & 0 & (+4) \\
\hline
 & 0 & 1 & 1 & 0 & +6 \checkmark
\end{array}
$$

$$
\begin{array}{rrrrrrl}
\cancel{\times} & & 1 & 1 & 1 & 0 & (-2) \\
+ & & 1_{1} & 1 & 0 & 0 & (-4) \\
\hline
 & & 1 & 0 & 1 & 0 &
\end{array}
$$

$= -(-1010) = -(0101+1)$

$= -(0110) = -6$

$$
\begin{array}{rrrrrl}
 & 0 & 0 & 1 & 0 & (+2) \\
+ & 1 & 1 & 0 & 0 & (-4) \\
\hline
 & 1 & 1 & 1 & 0 &
\end{array}
$$

$= -(-1110) = -(0001+1)$

$= -(0010) = -2$

$$
\begin{array}{rrrrrrl}
\cancel{\times} & & 1 & 1 & 1 & 0 & (-2) \\
+ & & 0_{1} & 1 & 0 & 0 & (+4) \\
\hline
 & & 0 & 0 & 1 & 0 & +2 \checkmark
\end{array}
$$

Subtraction in Two's Complement

- $A - B = A + (-B) = A + \overline{B} + 1$

- 0010 - 0110

$0010 + (-0110) = 0010 + (1001 + 1) = 0010 + 1010$

$2 \quad -6$

$$\begin{array}{r} 0010 \\ +\,1010 \\ \hline 1100 \end{array}$$

$= 1100$

$-(0011 + 1) = -0100$

$= -4$

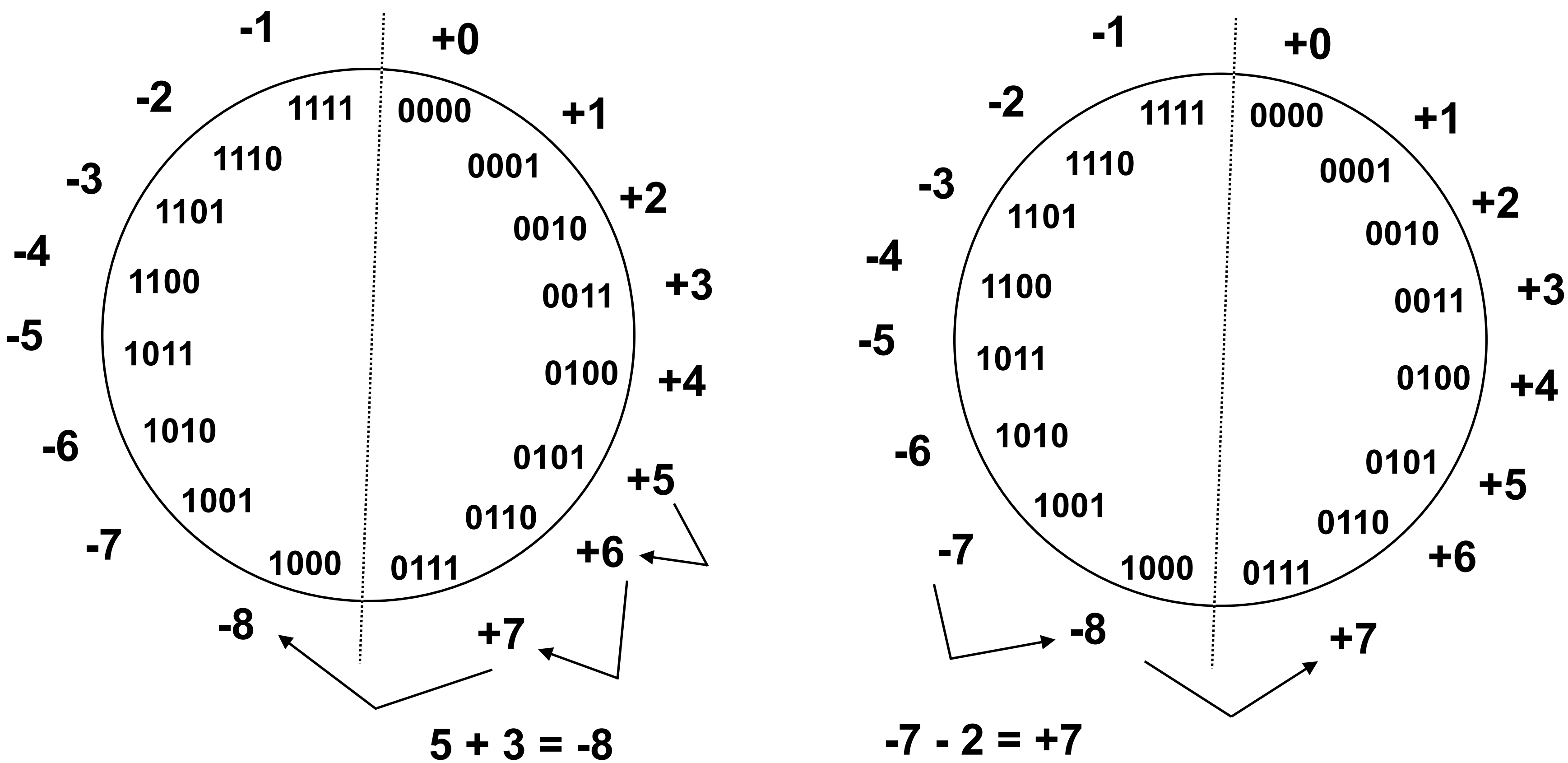
- 1011 - 1001

- 1011 - 0001

Overflows in Two's Complement

Add two positive numbers but get a negative number

or two negative numbers but get a positive number



Overflow Detection in Two's Complement

5	0 1 0 1
3	0 0 1 1
-8	1 0 0 0

Overflow

-7	1 0 0 1
-2	1 1 1 0
7	0 1 1 1

Overflow

5	0 1 0 1
2	0 0 1 0
7	0 1 1 1

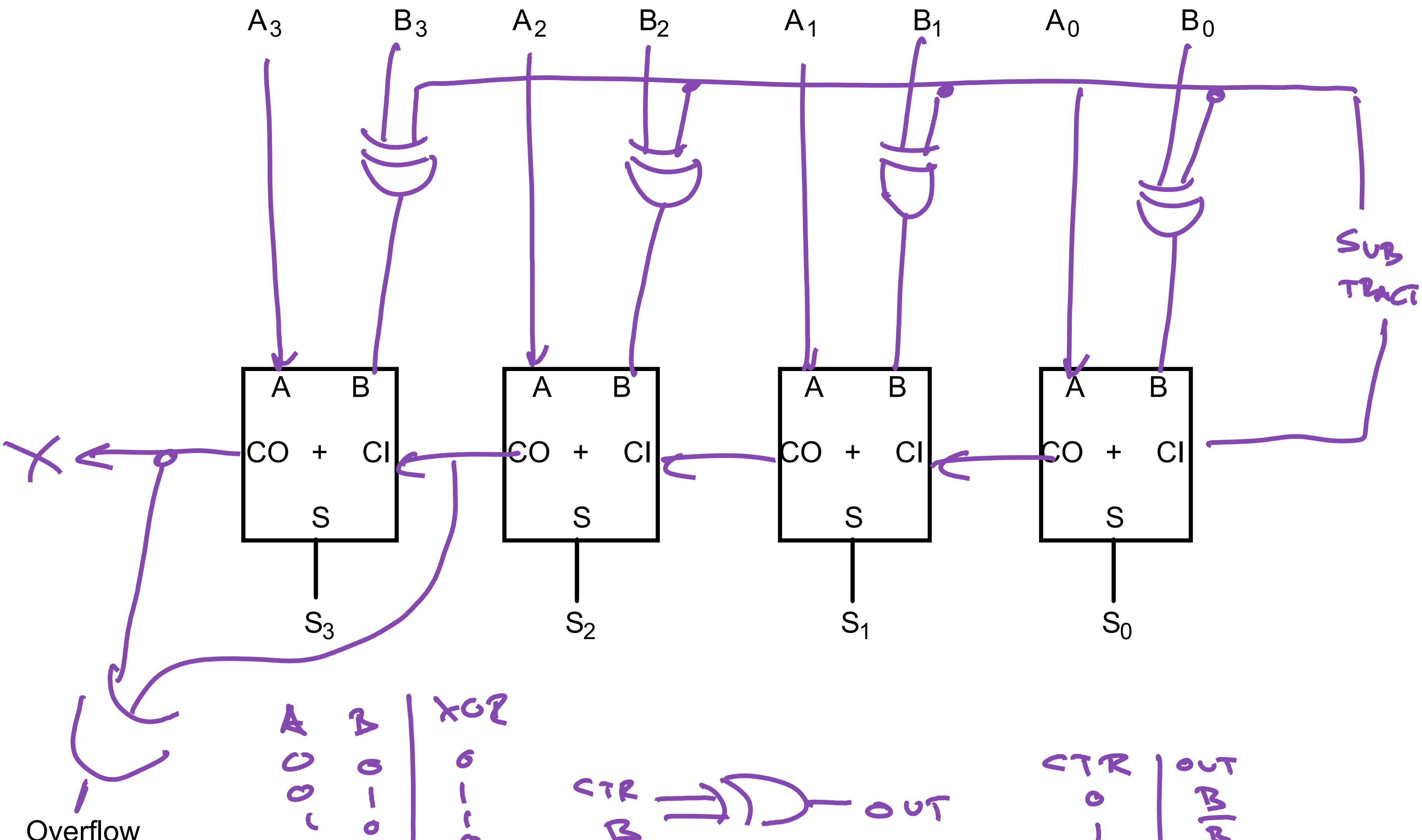
No overflow

-3	1 1 0 1
-5	1 0 1 1
-8	1 0 0 0

No overflow

OVERFLOW = $C_{IN} \oplus C_{OUT}$ OF HIGHEST ORDER NMBR.

Adder/Subtractor



A	B	XOR
0	0	0
0	1	1
1	0	1
1	1	0

CTR	OUT
0	B
1	$\overline{B}$

A - B = A + (-B) = A + $\overline{\text{B}}$ + 1

Converting Decimal to Two's Complement

- Convert absolute value to unsigned binary, then fixed width, then negate if necessary

- Convert $(-9)_{10}$ to 6-bit Two's Complement

- Convert $(9)_{10}$ to 6-bit Two's Complement

Converting Two's Complement to Decimal

- If Positive, convert as normal;
  If Negative, negate then convert.

- Convert $(11010)_2$ to Decimal

- Convert $(01101)_2$ to Decimal

Sign Extension

- To convert from N-bit to M-bit Two's Complement (N<M), simply duplicate sign bit:
- Convert $(0010)_2$ to 8-bit Two's Complement
- Convert $(1011)_2$ to 8-bit Two's Complement